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Thermodynamics of Two-Component Log-Gases with Alternating Charges

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Abstract We consider a one-dimensional gas of positive and negative unit charges interacting via a logarithmic potential, which is in thermal equilibrium at the (dimensionless) inverse temperature β . In a previous paper [Šamaj, L.: J. Stat. Phys. **105**, 173–191 (2001)], the exact thermodynamics of the unrestricted log-gas of pointlike charges was obtained using an equivalence with a $(1+1)$ -dimensional boundary sine-Gordon model. The present aim is to extend the exact study of the thermodynamics to the log-gas on a line with alternating \pm charges. The formula for the ordered grand partition function is obtained by using the exact results of the Thermodynamic Bethe ansatz. The complete thermodynamics of the ordered log-gas with pointlike charges is checked by a small- β expansion and at the collapse point $\beta_c = 1$. The inclusion of a small hard core around particles permits us to go beyond the collapse point. The differences between the unconstrained and ordered versions of the log-gas are pointed out.

Keywords Two-component log-gas · Charge ordering · Exact thermodynamics · Thermodynamic Bethe ansatz

1 Introduction

We study thermal equilibrium properties of a symmetric two-component plasma (Coulomb gas) which consists of mobile pointlike positive and negative unit charges ± 1 , confined in a two-dimensional (2D) domain of points $\mathbf{r} = (x, y)$. The charges are immersed in a homogeneous medium of dielectric constant 1, the system as a whole is electroneutral. In Gauss units, the Coulomb potential $\phi(\mathbf{r})$, induced by a unit charge at the origin $\mathbf{0}$, is given by

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the 2D Poisson equation

$$\Delta\phi(\mathbf{r}) = -2\pi\delta(\mathbf{r}). \quad (1.1)$$

Such definition of the 2D Coulomb potential maintains many generic properties of “realistic” three-dimensional Coulomb systems with $1/r$ potential, e.g. the screening sum rules [25]. In an infinite space, the solution of Eq. (1.1) reads

$$\phi(\mathbf{r}) = -\ln\left(\frac{r}{r_0}\right), \quad (1.2)$$

where $r = |\mathbf{r}|$ and r_0 is a length scale which fixes the zero point of the potential. The interaction energy of two charges q and q' at the respective spatial positions \mathbf{r} and \mathbf{r}' is equal to $v_{qq'}(\mathbf{r}, \mathbf{r}') = qq'\phi(|\mathbf{r} - \mathbf{r}'|)$. The corresponding Boltzmann factor at the (dimensionless) inverse temperature $\beta = 1/(k_B T)$ reads $\exp[-\beta v_{qq'}(\mathbf{r}, \mathbf{r}')] = |\mathbf{r} - \mathbf{r}'|^{\beta qq'}$. For two oppositely charged pointlike species at distance r , the Boltzmann factor $r^{-\beta}$ is integrable at small distances $r \rightarrow 0$ for small enough β , $\beta < \beta_c$. Thus, the thermodynamics of pointlike charges on a continuum space is well defined only in the high-temperature region $\beta < \beta_c$. For low temperatures $\beta \geq \beta_c$, the short-distance (ultraviolet) collapse of oppositely charged point particles makes the thermodynamics unstable; the thermodynamic stability is restored by considering a short-distance regularization of the Coulomb potential, i.e. via hard cores attached to particles which prevent them from touching one another. The value of the collapse (inverse) temperature β_c depends on the dimensionality of the compact domain A to which the charge system is constrained.

In the case of a 2D domain A , the 2D integral $\int_A d^2r |r|^{-\beta}$ is finite at small distances provided that $\beta < 2$, i.e. $\beta_c = 2$; there is no problem at large $|r|$ because the interaction is screened by the conducting system. A typical domain of this kind is an infinite space, $A \rightarrow \mathbb{R}^2$, which defines the bulk 2D Coulomb gas. The complete thermodynamics of the system of pointlike \pm charges in the stability region $\beta < 2$ was derived in [26]; for a short review of exact results for bulk and surface thermodynamics, together with asymptotic large-distance behavior of charge and density correlation functions, see [28]. The derivation was based on an equivalence between the 2D Coulomb gas and the (1+1)-dimensional sine-Gordon theory, transferring from that integrable theory some exact results obtained within the Thermodynamic Bethe ansatz (TBA). The extension of the exact thermodynamics of the bulk Coulomb gas beyond the collapse point $\beta_c = 2$, with a hard-core regularization of the Coulomb interaction and in the region of low particle densities, was proposed in [23]. An electroneutrality sum rule and the leading short-distance behavior of pair correlation functions was used to go up to $\beta = 3$. Applying a systematic short-distance expansion of correlation functions [31], Téllez showed [32] that one can proceed, in principle, up to the Kosterlitz-Thouless transition of an infinite order from a high-temperature conductor phase (a non-zero fraction of the positive and negative charges is dissociated) to a low-temperature insulator phase (the positive and negative charges form neutral pairs) which takes place at $\beta_{KT} = 4$ for low densities.

In the case of a one-dimensional (1D) domain A , say the infinite line $x \in (-\infty, \infty)$, the 1D integral $\int_A dx x^{-\beta}$ is finite at small distances provided

that $\beta < 1$, i.e. $\beta_c = 1$. Such systems, to which we refer as log-gases, have evoked much of interest because of their relationship to various models of condensed matter. There exist two basic versions of two-component log-gases: without and with space restriction on the ordering of \pm charges.

- **Unconstrained log-gas:** The “standard” 1D log-gas without any space restriction on the \pm charges is related to dissipative quantum mechanics [29, 5, 20, 6] and to the problem of non-equilibrium quantum transport through a point contact in a 1D Luttinger liquid [24, 11]. The lattice version of the model, which represents a kind of short-distance regularization of the Coulomb potential, was exactly solved (the grand partition function and the particle correlation functions) at $\beta = 1$ [15] and $\beta = 2, 4$ [16, 17]. The conductor-insulator phase diagram was conjectured in [18]; various approaches indicate that the Kosterlitz-Thouless phase transition should occur at $\beta_{KT} = 2$, independently of the particle density. Thermodynamic properties of the continuous version of the model with pointlike charges was solved in the whole stability region $\beta < 1$ in [27], by exploring the TBA results for an equivalent $(1+1)$ -dimensional boundary sine-Gordon theory [10, 11, 4, 9].
- **Log-gas with charge ordering:** The 1D log-gas with \pm charges required to alternate in space is equivalent to the Kondo problem with spin- $\frac{1}{2}$ impurity [2, 3, 30]. The lattice version of the system is exactly solvable at the “collapse” isotherm $\beta = 1$ [15]. The charge ordered system is expected to exhibit a dielectric phase at an arbitrary temperature [14].

The asymptotic large-distance behavior of the particle correlation functions was studied in detail for both versions of the 1D two-component log-gas in [1].

The aim of the present work is to extend the exact study of the thermodynamics to the two-component log-gas on a line with alternating \pm charges. The basic formula for the ordered grand partition function is obtained by using the TBA results [12, 13], associated with a fusion relation between the grand partition functions of the unconstrained and ordered log-gases. The complete thermodynamics of the ordered log-gas with pointlike charges is derived. The results are checked by a small- β expansion and at the collapse point $\beta_c = 1$. The inclusion of small hard cores around particles permits us to extend the thermal analysis beyond the collapse point $\beta_c = 1$. The important differences between the unconstrained and ordered versions of the log-gas are pointed out.

The paper is organized as follows.

We start with a brief recapitulation and an extension of the results for the unconstrained 1D log-gas [27] in Sect. 2. Sect. 2.1 deals with the case of pointlike charges. The inclusion of a small hard core around particles, which permits us to study the thermodynamics beyond the collapse point $\beta_c = 1$, is the subject of Sect. 2.2. Here, we apply two methods: the one based on a perfect screening sum rule valid for conducting systems and the other, more general, based on the explicit definition of the grand partition function.

The thermodynamics of the ordered 1D log-gas is derived in Sect. 3. As in the previous unconstrained case, the system of pointlike charges is treated in Sect. 3.1 and the inclusion of the hard core to particles, via the definition

of the grand partition function, is worked out in Sect. 3.2. The differences between the unconstrained and ordered versions of the log-gas are pointed out.

Sect. 4 is Conclusion.

Auxiliary calculations are shifted aside to Appendices. A fusion relation between the grand partition functions of the unconstrained and ordered log-gases is used to rederive in an alternative way a relationship between the corresponding bulk pressures in Appendix A. The small- β expansion of the obtained density-fugacity relationship for the ordered log-gas is checked by microscopic calculations in Appendix B.

2 Extension of the results for the unconstrained log-gas

First we recapitulate and extend the exact results for the thermodynamics of the unconstrained log-gas obtained in [27]. We start with pointlike charges. The inclusion of hard cores to particles, which will allow us to pass through the collapse point $\beta_c = 1$, is described in the subsequent part.

2.1 Pointlike particles

We go to an infinite line through the circle of radius R and circumference $L = 2\pi R$, taking at the end the limit $R \rightarrow \infty$. The position of a particle on the circle is specified by the angle $\varphi \in [0, 2\pi)$. Since the distance between particles at angle positions φ and φ' is $2R|\sin[(\varphi - \varphi')/2]|$, the 2D Coulomb potential takes the form

$$\phi(\varphi, \varphi') = -\ln \left[\frac{2R}{r_0} \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right| \right]. \quad (2.1)$$

Defining the line coordinate as $x = R\varphi$, the 1D realization of the potential (1.2), $\phi(x, x') = -\ln(|x - x'|/r_0)$, results from this expression as the $R \rightarrow \infty$ limit. The Boltzmann factor of two charges q and q' at the respective angle positions φ and φ' reads as

$$\exp[-\beta qq' \phi(\varphi, \varphi')] = \left| \frac{2R}{r_0} \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|^{\beta qq'}. \quad (2.2)$$

We shall work in the grand canonical ensemble. Due to the charge \pm symmetry, chemical potentials of the species can be taken equivalent, $\mu_+ = \mu_- = \mu$. Only neutral configurations with the particle numbers $N_+ = N_- = N$ are considered. The position angles of (+) charges will be denoted by φ_i and those of (−) charges by φ'_i ($i = 1, \dots, N$). The grand partition function is defined as

$$\Xi_L(\mu) = \sum_{N=0}^{\infty} \frac{\exp(2\beta\mu N)}{(N!)^2} \int_0^{2\pi} \frac{d\varphi_1 R}{\lambda} \int_0^{2\pi} \frac{d\varphi'_1 R}{\lambda} \dots \int_0^{2\pi} \frac{d\varphi_N R}{\lambda} \int_0^{2\pi} \frac{d\varphi'_N R}{\lambda}$$

$$\times \left| \frac{\prod_{(i<j)=1}^N \left[\frac{2R}{r_0} \sin \left(\frac{\varphi_i - \varphi_j}{2} \right) \right] \left[\frac{2R}{r_0} \sin \left(\frac{\varphi'_i - \varphi'_j}{2} \right) \right]}{\prod_{i,j=1}^N \left[\frac{2R}{r_0} \sin \left(\frac{\varphi_i - \varphi'_j}{2} \right) \right]} \right|^\beta, \quad (2.3)$$

where λ is the de Broglie wavelength and the dependence on the inverse temperature β is omitted. We introduce the rescaled fugacity $z_+ = z_- = z$,

$$z = \exp(\beta\mu) \frac{r_0^{\beta/2}}{\lambda}, \quad (2.4)$$

which has the dimension $[\text{length}]^{\frac{\beta}{2}-1}$. Factorizing out quantities of nonzero dimensions in (2.3), we end up with the representation

$$\Xi_L(z) = 1 + \sum_{N=1}^{\infty} \left[(2\pi)^{\beta/2} z L^{1-\frac{\beta}{2}} \right]^{2N} I_{2N}, \quad (2.5)$$

where the dimensionless configuration integrals

$$I_{2N} = \frac{1}{(N!)^2} \int_0^{2\pi} \frac{d\varphi_1}{2\pi} \int_0^{2\pi} \frac{d\varphi'_1}{2\pi} \dots \int_0^{2\pi} \frac{d\varphi_N}{2\pi} \int_0^{2\pi} \frac{d\varphi'_N}{2\pi} B_{2N}(\{\varphi_i\}, \{\varphi'_i\}) \quad (2.6)$$

involve the Boltzmann weights

$$B_{2N}(\{\varphi_i\}, \{\varphi'_i\}) = \left| \frac{\prod_{(i<j)=1}^N \left[2 \sin \left(\frac{\varphi_i - \varphi_j}{2} \right) \right] \left[2 \sin \left(\frac{\varphi'_i - \varphi'_j}{2} \right) \right]}{\prod_{i,j=1}^N \left[2 \sin \left(\frac{\varphi_i - \varphi'_j}{2} \right) \right]} \right|^\beta. \quad (2.7)$$

In the large- L limit, we pass from a finite circle to an infinite 1D line, $x \in (-\infty, \infty)$. According to elementary thermodynamics, the quantity $\ln \Xi_L(z)$ is expected to be extensive. In particular, the bulk pressure of the charge system $P(z)$, defined by

$$\beta P(z) = \lim_{L \rightarrow \infty} \frac{1}{L} \ln \Xi_L(z), \quad (2.8)$$

has a well-defined finite value. The only expansion parameter in the series representation (2.5) is the dimensionless combination $z L^{1-\frac{\beta}{2}}$, so that $\beta P(z) \propto z^{\frac{2}{2-\beta}}$. The proportionality β -dependent factor was found in [27] by using the TBA results obtained for an equivalent boundary sine-Gordon model [10, 11, 4, 9]. We take the TBA results from [10]; the notation in that work is related to ours as follows: $g = \beta/2$, $t = 1/g = 2/\beta$, $q = \exp(i\pi g) = \exp(i\pi\beta/2)$, $T = 1/L$ and a scale $T_B = 1/L_B$. The expansion parameter in (2.5) was expressed in terms of the length scale L_B as

$$z(2\pi)^{\beta/2} L^{1-\frac{\beta}{2}} = \Gamma\left(\frac{\beta}{2}\right) \left[\frac{L}{L_B} \frac{\Gamma\left(\frac{1}{2-\beta}\right)}{2\sqrt{\pi}\Gamma\left(\frac{\beta}{2(2-\beta)}\right)} \right]^{1-\frac{\beta}{2}}, \quad (2.9)$$

where Γ is the Gamma function [19]. For large L , the grand partition function is given by

$$\Xi_L \underset{L \rightarrow \infty}{\sim} \sqrt{\frac{\beta}{2}} \exp \left[\frac{L}{L_B} \frac{1}{2 \cos \left(\frac{\pi \beta}{2(2-\beta)} \right)} \right]. \quad (2.10)$$

Eliminating L_B from the couple of equations and using formulas for the Gamma functions [19]

$$\Gamma(1-y)\Gamma(y) = \frac{\pi}{\sin(\pi y)}, \quad \Gamma\left(\frac{1}{2}-y\right)\Gamma\left(\frac{1}{2}+y\right) = \frac{\pi}{\cos(\pi y)} \quad (2.11)$$

with real y , we finally arrive at the bulk pressure

$$\beta P(z) = \frac{1}{2\pi^{3/2}} \Gamma\left(\frac{1-\beta}{2-\beta}\right) \Gamma\left(\frac{\beta}{2(2-\beta)}\right) \left[\frac{2\pi z}{\Gamma(\beta/2)} \right]^{\frac{2}{2-\beta}}. \quad (2.12)$$

The one-body densities are defined as thermal averages

$$n_q(x) \equiv n_q = \left\langle \sum_j \delta_{q,q_j} \delta(x - x_j) \right\rangle, \quad (2.13)$$

where $\delta_{q,q'}$ is the Kronecker symbol, $\delta(x - x')$ the Dirac delta function and the index j runs over all particles. For the considered charge \pm symmetry, the species densities are equal to one another: $n_+ = n_- = n/2$, where the total number density of particles n is given by

$$n(z) = z \frac{\partial}{\partial z} \beta P(z). \quad (2.14)$$

Thus the density-fugacity relationship reads as

$$\frac{n}{z^{\frac{2}{2-\beta}}} = \frac{1}{\pi^{3/2}(2-\beta)} \Gamma\left(\frac{1-\beta}{2-\beta}\right) \Gamma\left(\frac{\beta}{2(2-\beta)}\right) \left[\frac{2\pi}{\Gamma(\beta/2)} \right]^{\frac{2}{2-\beta}}. \quad (2.15)$$

The small- β expansion of the rhs of this formula

$$\frac{n}{z^{\frac{2}{2-\beta}}} = 2\beta^{\frac{\beta}{2-\beta}} \exp \left([C + \ln(2\pi)] \frac{\beta}{2} + O(\beta^2) \right), \quad (2.16)$$

where C is the Euler constant, was checked in Appendix of [27] by using a renormalized Mayer expansion. Note that this expansion is non-analytic due to the appearance of the term $\beta^{\frac{\beta}{2-\beta}}$. On the other hand, the series in the exponential is analytic in β .

For a fixed z and in the limit $\beta \rightarrow 1^-$, the term $\Gamma((1-\beta)/(2-\beta)) \sim 1/(1-\beta)$ implies that the particle density n exhibits the expected collapse singularity

$$n \underset{\beta \rightarrow 1^-}{\sim} \frac{4z^2}{1-\beta}. \quad (2.17)$$

This singular behavior can be deduced indirectly by using a perfect screening sum rule for the one-body densities of the conducting system [25],

$$n_q = \int dx [U_{q,-q}(x) - U_{q,q}(x)], \quad (2.18)$$

where the Ursell functions are defined by

$$U_{q,q'}(x, x') \equiv U_{q,q'}(|x - x'|) = \left\langle \sum_{j \neq k} \delta_{q,q_j} \delta(x - x_j) \delta_{q',q_k} \delta(x' - x_k) \right\rangle - n_q n_{q'}. \quad (2.19)$$

The short-distance behavior of the Ursell function for a positive-negative pair of charges is given by the Boltzmann factor of the pair interaction [22,21],

$$U_{q,-q}(x) \underset{x \rightarrow 0}{\sim} z^2 |x|^{-\beta}. \quad (2.20)$$

For $\beta \rightarrow 1^-$, the integral in (2.18) is dominated by this short-distance behavior and we have

$$\frac{n}{2} \sim \int_{-\ell}^{\ell} dx \frac{z^2}{|x|^\beta} = \frac{2z^2}{1-\beta} \ell^{1-\beta} = \frac{2z^2}{1-\beta} + O(1), \quad (2.21)$$

where ℓ is a screening length of the Coulomb potential. This derivation of the singular behavior (2.17) points out the two-body nature of the collapse phenomenon.

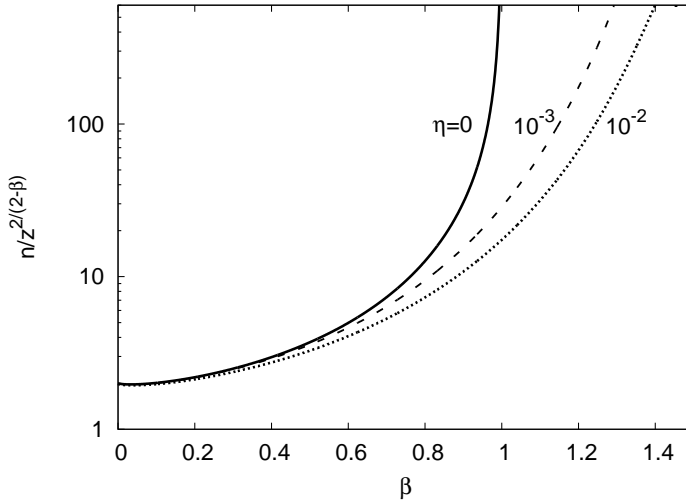


Fig. 1 The density-fugacity relationship versus the inverse temperature β for the unconstrained log-gas. The packing fraction $\eta = n\sigma$, where σ is the hard-core diameter. The solid curve describes pointlike charges ($\eta = 0$). The dashed and dotted curves correspond to $\eta = 10^{-3}$ and 10^{-2} , respectively.

The plot of the density-fugacity relationship for pointlike particles (2.15), well defined up to $\beta_c = 1$, is represented in Fig. 1 by the solid curve. Although the function looks like monotonously increasing, this is not true: as is seen in Fig. 2, the function decreases and reaches the minimum for small β .

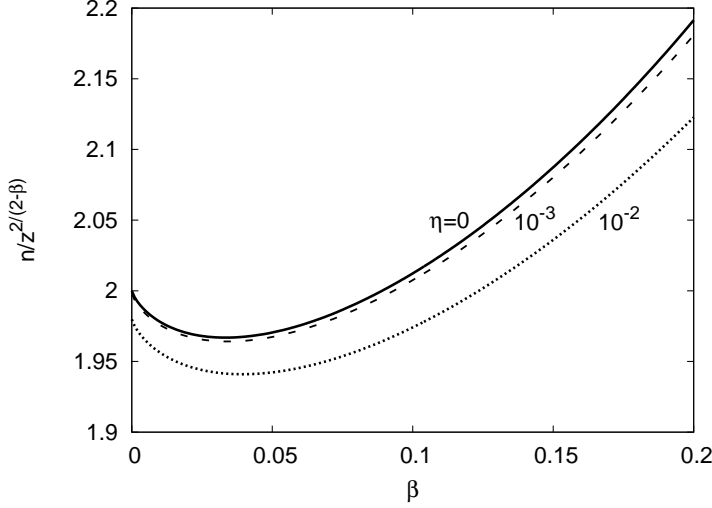


Fig. 2 Fig. 1 on a smaller scale to document the non-monotonous behavior of the density-fugacity plot for small β .

The virial equation of state, which relates the pressure P and the particle density n , takes a simple form

$$\beta P = \left(1 - \frac{\beta}{2}\right) n \quad (2.22)$$

which is characteristic for log-gas systems.

Having the density-fugacity relationship, the full thermodynamics is obtained by passing from the grand-canonical to the canonical ensemble via the Legendre transformation

$$F_L(\beta, N) = -\frac{1}{\beta} \ln \Xi_L + \mu N, \quad (2.23)$$

where F_L is the Helmholtz free energy and $N = nL$. The (dimensionless) specific free energy, defined as $f \equiv \beta F_L/N$, thus reads

$$\begin{aligned} f(\beta, n) = & \ln(\lambda n) - \frac{\beta}{2} \ln(2r_0 n) + \frac{1}{2} \left(1 - \frac{3\beta}{2}\right) \ln \pi \\ & - \left(1 - \frac{\beta}{2}\right) \left[1 - \ln\left(1 - \frac{\beta}{2}\right)\right] + \ln \Gamma\left(\frac{\beta}{2}\right) \\ & - \left(1 - \frac{\beta}{2}\right) \left[\ln \Gamma\left(\frac{1-\beta}{2-\beta}\right) + \ln \Gamma\left(\frac{\beta}{2(2-\beta)}\right)\right]. \end{aligned} \quad (2.24)$$

The excess (i.e. over ideal) internal energy per particle and specific heat at constant volume are given by

$$u^{\text{ex}} = \frac{\partial}{\partial \beta} f(\beta, n), \quad \frac{c_v^{\text{ex}}}{k_B} = -\beta^2 \frac{\partial^2}{\partial \beta^2} f(\beta, n), \quad (2.25)$$

respectively. For the specific heat, we obtain explicitly

$$\begin{aligned} \frac{c_v^{\text{ex}}}{k_B} = & \frac{\beta^2}{2(2-\beta)^3} \left[\psi' \left(\frac{1-\beta}{2-\beta} \right) + \psi' \left(\frac{\beta}{2(2-\beta)} \right) \right] \\ & - \frac{\beta^2}{2(2-\beta)} - \frac{\beta^2}{4} \psi' \left(\frac{\beta}{2} \right), \end{aligned} \quad (2.26)$$

where the psi function and its derivative are given by

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x), \quad \psi'(x) = \sum_{j=0}^{\infty} \frac{1}{(x+j)^2}. \quad (2.27)$$

The specific heat exhibits the Laurent series expansion around the collapse point $\beta \rightarrow 1^-$,

$$\frac{c_v^{\text{ex}}}{k_B} = \frac{1}{2(1-\beta)^2} - \frac{3}{2(1-\beta)} + O(1). \quad (2.28)$$

2.2 Particles with hard cores

To go beyond the collapse point $\beta_c = 1$, we have to consider a short-distance regularization of the interaction Coulomb potential, especially for oppositely charged particles. We introduce a hard core of diameter σ around each particle which does not allow the charges to touch one another:

$$v_{q,q'}(x, x') = \begin{cases} -qq' \ln(|x - x'|/r_0) & \text{if } |x - x'| > \sigma, \\ \infty & \text{if } |x - x'| \leq \sigma. \end{cases} \quad (2.29)$$

The total particle density now depends also on σ , $n(z, \sigma)$. The hard-core diameter σ , or more precisely the dimensionless combination

$$\xi = \sigma z^{\frac{2}{2-\beta}}, \quad (2.30)$$

will be assumed to be small, $\xi \rightarrow 0$, and we shall look for the leading hard-core correction to the pointlike system.

We first derive the leading hard-core correction in analogy with the derivation for the 2D Coulomb gas presented in [23]. The unconstrained log-gas system is in its conducting phase up to $\beta_{KT} = 2$, so that the screening sum rule (2.18) holds and the Ursell functions are well defined also beyond the collapse point. The difference $U_{q,-q} - U_{q,q}$ vanishes inside the hard core for the potential (2.29), and the total particle number density is given by

$$n(z, \sigma) = 2 \int_{\sigma}^{\infty} dx [U_{q,-q}(x; z, \sigma) - U_{q,q}(x; z, \sigma)]. \quad (2.31)$$

We can write

$$U_{q,q'}(x; z, \sigma) = U_{q,q'}(x; z, 0) + \Delta_{q,q'}(x; z, \sigma), \quad x > \sigma, \quad (2.32)$$

which defines $\Delta_{q,q'}(x; z, \sigma)$, vanishing when $\sigma \rightarrow 0$, as the change of the Ursell function due to the introduction of the hard core σ to pointlike particles. Subtraction of Eq. (2.31) with $\sigma > 0$ and the same equation with $\sigma = 0$ leads to

$$\begin{aligned} n(z, \sigma) - n(z, 0) = & -2 \int_0^\sigma dx [U_{q,-q}(x; z, 0) - U_{q,q}(x; z, 0)] \\ & + 2 \int_\sigma^\infty dx [\Delta_{q,-q}(x; z, \sigma) - \Delta_{q,q}(x; z, \sigma)]. \end{aligned} \quad (2.33)$$

We make a heuristic assumption: in the small- σ limit, we can neglect the quantities $\Delta_{q,\pm q}$ in (2.33) as they contribute to subleading hard-core corrections. Such assumption has already been made in [7, 8, 23] for the 2D Coulomb gas. Consequently,

$$n(z, \sigma) = n(z, 0) - 2 \int_0^\sigma dx [U_{q,-q}(x; z, 0) - U_{q,q}(x; z, 0)]. \quad (2.34)$$

As before, the integral is dominated by the short-distance behavior of the Ursell function (2.20) and we arrive at the basic result

$$n(z, \sigma) = n(z, 0) - 4z^2 \frac{\sigma^{1-\beta}}{1-\beta}, \quad \xi \rightarrow 0. \quad (2.35)$$

Strictly speaking, this result was derived in the stability region $0 \leq \beta < 1$ where $n(z, 0)$ is well defined by (2.15). It is however reasonable to assume that the formula (2.35) can be analytically continued beyond $\beta_c = 1$ because both the sum rule (2.31) and the leading short-distance behavior (2.20), which were crucial in its derivation, remain valid up to $\beta_{KT} = 2$.

To provide a convenient representation of (2.35), we express the density-fugacity relation for pointlike particles (2.15) in the form

$$n(z, 0) = \frac{4\Phi(\beta)}{1-\beta} z^{\frac{2}{2-\beta}}, \quad (2.36)$$

where the function

$$\Phi(\beta) = \frac{1}{4\pi^{3/2}} \Gamma\left(\frac{3-2\beta}{2-\beta}\right) \Gamma\left(\frac{\beta}{2(2-\beta)}\right) \left[\frac{2\pi}{\Gamma(\beta/2)}\right]^{\frac{2}{2-\beta}} \quad (2.37)$$

results from the equality $\Gamma(x+1) = x\Gamma(x)$ for $x = (1-\beta)/(2-\beta)$. $\Phi(\beta)$ was chosen such that it equals to 1 at the collapse point, namely

$$\Phi(\beta) = 1 + [C + \ln(4\pi)](\beta - 1) + O((\beta - 1)^2). \quad (2.38)$$

Using the representation (2.36) in (2.35), we get the density-fugacity relationship which involves the leading hard-core correction:

$$\frac{n(z, \sigma)}{z^{\frac{2}{2-\beta}}} = \frac{4}{1-\beta} [\Phi(\beta) - \xi^{1-\beta}], \quad \xi \rightarrow 0. \quad (2.39)$$

The leading correction $\propto \xi^{1-\beta}$ is marginal in the stability region $0 \leq \beta < 1$. It makes the density finite at the collapse point $\beta_c = 1$,

$$\frac{n(z, \sigma)}{z^2} = -4 [\ln \xi + C + \ln(4\pi)]. \quad (2.40)$$

Note that the density is positive for small ξ and diverges for $\xi \rightarrow 0$, as it should be. The leading correction becomes relevant in the region $\beta \geq 1$, up to $\beta = 3/2$ at which $\Phi(\beta)$ diverges. To overcome this divergence, we have to account for higher-order terms of the short-distance expansion of the Ursell functions in the screening sum rule (2.34), in close analogy with [23, 31, 32]. Such analysis goes beyond the scope of this work.

It is convenient to pass from the parameter ξ to the packing fraction

$$\eta = n\sigma. \quad (2.41)$$

Since each particle occupies the length σ , it must hold that $L/N \geq \sigma$, i.e. $\eta \leq 1$. Denoting $f \equiv n(z, \sigma)/z^{\frac{2}{2-\beta}}$, we have $\eta = f\xi$. Multiplying both sides of Eq. (2.39) by $(1 - \beta)f^{1-\beta}$, we get

$$(1 - \beta)f^{2-\beta} - 4 [\Phi(\beta)f^{1-\beta} - \eta^{1-\beta}] = 0. \quad (2.42)$$

At $\beta = 0$, from two solutions of this equation we take the one $f = 1 + \sqrt{1 - 4\eta}$ and follow this branch when increasing β . The density-fugacity relation for the packing fractions 10^{-3} and 10^{-2} are plotted in Figs. 1 and 2 by the dashed and dotted curves, respectively.

For particles with hard cores, in analogy with (2.14) it holds

$$n(z, \sigma) = z \frac{\partial}{\partial z} \beta P(z, \sigma). \quad (2.43)$$

With regard to (2.35), we obtain the equation of state

$$\beta P(z, \sigma) = \beta P(z, 0) - 2z^2 \frac{\sigma^{1-\beta}}{1-\beta}, \quad \xi \rightarrow 0, \quad (2.44)$$

where $\beta P(z, 0)$ is given by (2.12). This relation can be deduced directly from the definition of the grand partition function; the importance of such derivation is evident for non-conducting systems which do not exhibit screening. As the collapse phenomenon does not involve $(++)$ and $(--)$ pairs of charges, we simplify the model by considering hard cores only between oppositely charged particles. In particular, $v_{qq}(x, x') = -\ln(|x - x'|/r_0)$ for an arbitrary distance between particles and

$$v_{q,-q}(x, x') = \begin{cases} \ln(|x - x'|/r_0) & \text{if } |x - x'| > \sigma, \\ \infty & \text{if } |x - x'| \leq \sigma. \end{cases} \quad (2.45)$$

With the definition (2.4) of the fugacity z , the grand partition function of this neutral 1D system of size L (which is large enough to neglect boundary effects, but not infinite) is given by

$$\Xi_L(z, \sigma) = 1 + \sum_{N=1}^{\infty} z^{2N} I_{2N}(\sigma), \quad (2.46)$$

where the configuration integrals

$$I_{2N}(\sigma) = \frac{1}{(N!)^2} \int_0^L dx_1 \int_0^L dx'_1 \cdots \int_0^L dx_N \int_0^L dx'_N \\ \times B_{2N}(\{x_i\}, \{x'_i\}) \prod_{i,j=1}^N \theta(|x_i - x'_j| - \sigma) \quad (2.47)$$

involve the Boltzmann weights

$$B_{2N}(\{x_i\}, \{x'_i\}) = \left| \frac{\prod_{(i<j)=1}^N (x_i - x_j)(x'_i - x'_j)}{\prod_{i,j=1}^N (x_i - x'_j)} \right|^\beta. \quad (2.48)$$

Here, $\theta(x)$ is the Heaviside theta function:

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases} \quad (2.49)$$

which fulfills the obvious relation $\theta(x) + \theta(-x) = 1$. Our trick consists in substituting each theta function in (2.47) by

$$\theta(|x_i - x'_j| - \sigma) = 1 - \theta(\sigma - |x_i - x'_j|). \quad (2.50)$$

For the lowest-order integral $I_2(\sigma)$, we get

$$I_2(\sigma) = I_2(0) - \frac{1}{(1!)^2} \int_0^L dx_1 \int_0^L dx'_1 \frac{1}{|x_1 - x'_1|^\beta} \theta(\sigma - |x_1 - x'_1|) \\ \sim I_2(0) - 2L \int_0^\sigma dx x^{-\beta} = I_2(0) - 2L \frac{\sigma^{1-\beta}}{1-\beta}. \quad (2.51)$$

For a general integral $I_{2N}(\sigma)$, we expand the product of theta functions in (2.47) as follows

$$\prod_{i,j=1}^N \theta(|x_i - x'_j| - \sigma) = 1 - \sum_{i,j=1}^N \theta(\sigma - |x_i - x'_j|) \\ + \sum_{\substack{i,j,k,l=1 \\ (i,j) \neq (k,l)}}^N \theta(\sigma - |x_i - x'_j|) \theta(\sigma - |x_k - x'_l|) + \cdots. \quad (2.52)$$

The first term on the rhs (unity) gives $I_{2N}(0)$. The second term contains N^2 summands, each provides the same contributions because the Boltzmann weight (2.48) is symmetric with respect to any interchange of two x_i or two x'_i . This is why we can substitute the sum by $N^2 \theta(\sigma - |x_1 - x'_1|)$. The oppositely charged particles at x_1 and x'_1 have to be very close to one another for small σ , and therefore form an almost neutral entity which decouples from all other charges. The integration over x_1 and x'_1 produces the previous factor $-2L\sigma^{1-\beta}/(1-\beta)$, while the integration over x -coordinates of all remaining

$2(N-1)$ charges, when multiplied by $N^2/(N!)^2$, implies $I_{2(N-1)}(0)$. The summands in the third term on the rhs of (2.52) can have $i = k$ or $j = l$; we exclude such terms because they describe the situation of one say (+) charge coupled to two (−) charges and the corresponding power $\sigma^{2-\beta}$ is marginal in comparison with $\sigma^{1-\beta}$. Then there is $[N(N-1)]^2/2!$ of equivalent terms of type $\theta(\sigma - |x_1 - x'_1|)\theta(\sigma - |x_2 - x'_2|)$. As before, the couples of particles with coordinates (x_1, x'_1) and (x_2, x'_2) form neutral entities for small σ , each contributing by the factor $-2L\sigma^{1-\beta}/(1-\beta)$. The integration over x -coordinates of all remaining $2(N-2)$ charges, multiplied by $[N(N-1)]^2/(2!N!^2)$, implies $I_{2(N-2)}(0)/2!$. Proceeding further in this way we end up with the recurrence relation

$$I_{2N}(\sigma) = I_{2N}(0) - \left(2L \frac{\sigma^{1-\beta}}{1-\beta}\right) I_{2(N-1)}(0) + \frac{1}{2!} \left(2L \frac{\sigma^{1-\beta}}{1-\beta}\right)^2 I_{2(N-2)}(0) + \dots \quad (2.53)$$

with $I_0(0) = 1$. Inserting this recurrence into the definition (2.46), we end up with

$$\Xi_L(z, \sigma) = \Xi_L(z, 0) \exp\left(-2z^2 \frac{\sigma^{1-\beta}}{1-\beta} L\right). \quad (2.54)$$

Applying the logarithm to both sides of this equation and using the definition of the pressure (2.8), we recover the needed result (2.44).

3 Thermodynamics of the ordered 1D log-gas

We proceed by the log-gas system with alternating \pm charges. To distinguish the quantities from the unconstrained case, we add to them the superscript “(ord)”, i.e. ordered. As before, we start with pointlike charges in Sect. 3.1 and then, in order to pass the collapse point $\beta_c = 1$, we attach to particles a small hard core in Sect. 3.2.

3.1 Pointlike particles

For a neutral configuration of $2N$ particles constrained to a circle of circumference L , we consider only ordered sequences with alternating \pm charges:

$$\varphi_1 > \varphi'_1 > \varphi_2 > \varphi'_2 \dots > \varphi_N > \varphi'_N. \quad (3.1)$$

The grand partition function is expressible as

$$\Xi_L^{(\text{ord})}(z) = 1 + \sum_{N=1}^{\infty} \left[(2\pi)^{\beta/2} z L^{1-\frac{\beta}{2}}\right]^{2N} I_{2N}^{(\text{ord})}, \quad (3.2)$$

where the configuration integrals

$$I_{2N}^{(\text{ord})} = \int_0^{2\pi} \frac{d\varphi_1}{2\pi} \int_0^{\varphi_1} \frac{d\varphi'_1}{2\pi} \int_0^{\varphi'_1} \frac{d\varphi_2}{2\pi} \int_0^{\varphi_2} \frac{d\varphi'_2}{2\pi} \dots \int_0^{\varphi_{N-1}} \frac{d\varphi_N}{2\pi} \int_0^{\varphi_N} \frac{d\varphi'_N}{2\pi} \times B_{2N}(\{\varphi_i\}, \{\varphi'_i\}) \quad (3.3)$$

involve the Boltzmann weights (2.7). We are interested in the thermodynamic limit of the pressure $P^{(\text{ord})}$ and the total particle density $n^{(\text{ord})}$,

$$\beta P^{(\text{ord})}(z) = \lim_{L \rightarrow \infty} \frac{1}{L} \ln \Xi_L^{(\text{ord})}(z), \quad n^{(\text{ord})}(z) = z \frac{\partial}{\partial z} \beta P^{(\text{ord})}(z). \quad (3.4)$$

As in the unconstrained case, both $\beta P^{(\text{ord})}(z)$ and $n^{(\text{ord})}(z)$ scale with the fugacity like $z^{\frac{2}{2-\beta}}$.

The ordered configuration integrals (3.3) can be compared to their unconstrained counterparts (2.6). The point is that the Boltzmann weights (2.7) are invariant with respect to any interchange of two φ_i or two φ'_i . Since there exist $N!$ possible ways how to order $\{\varphi_i\}$, and similarly for $\{\varphi'_i\}$, we can choose the special angle constraints

$$\varphi_1 > \varphi_2 > \dots > \varphi_{N-1} > \varphi_N, \quad \varphi'_1 > \varphi'_2 > \dots > \varphi'_{N-1} > \varphi'_N \quad (3.5)$$

and rewrite (2.6) as follows

$$I_{2N} = \int_0^{2\pi} \frac{d\varphi_1}{2\pi} \int_0^{2\pi} \frac{d\varphi'_1}{2\pi} \int_0^{\varphi_1} \frac{d\varphi_2}{2\pi} \int_0^{\varphi'_1} \frac{d\varphi'_2}{2\pi} \dots \int_0^{\varphi_{N-1}} \frac{d\varphi_N}{2\pi} \int_0^{\varphi'_{N-1}} \frac{d\varphi'_N}{2\pi} \times B_{2N}(\{\varphi_i\}, \{\varphi'_i\}). \quad (3.6)$$

The alternating configuration space (3.1) constitutes a subset of the reduced configuration space (3.5). Since the Boltzmann weights are positive, we conclude that $I_{2N} \geq I_{2N}^{(\text{ord})}$ or, equivalently, $\Xi_L(z) \geq \Xi_L^{(\text{ord})}(z)$. In the thermodynamic limit $L \rightarrow \infty$, we have the rigorous inequalities

$$\beta P(z) \geq \beta P^{(\text{ord})}(z), \quad n(z) \geq n^{(\text{ord})}(z). \quad (3.7)$$

We take the TBA results from [10]. The expansion parameter in (3.2) is expressible in terms of the length scale L_B in analogy with (2.9) if the substitution $z \rightarrow \tau z$ with $\tau = i/(q - q^{-1}) = 1/[2 \sin(\pi\beta/2)]$ is made:

$$\frac{z}{2 \sin\left(\frac{\pi\beta}{2}\right)} (2\pi)^{\beta/2} L^{1-\frac{\beta}{2}} = \Gamma\left(\frac{\beta}{2}\right) \left[\frac{L}{L_B} \frac{\Gamma\left(\frac{1}{2-\beta}\right)}{2\sqrt{\pi}\Gamma\left(\frac{\beta}{2(2-\beta)}\right)} \right]^{1-\frac{\beta}{2}}. \quad (3.8)$$

For large L , the grand partition function for the ordered system was found to behave as

$$\Xi_L^{(\text{ord})} \underset{L \rightarrow \infty}{\sim} \exp \left[\frac{L}{L_B} \tan \left(\frac{\pi\beta}{2(2-\beta)} \right) \right]. \quad (3.9)$$

Eliminating L_B from the two equations and using the formulas (2.11) for the Gamma functions, the bulk pressure is obtained in the form

$$\beta P^{(\text{ord})}(z) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1-\beta}{2-\beta}\right)}{\Gamma\left(\frac{4-3\beta}{4-2\beta}\right)} \left[\Gamma\left(1 - \frac{\beta}{2}\right) z \right]^{\frac{2}{2-\beta}}. \quad (3.10)$$

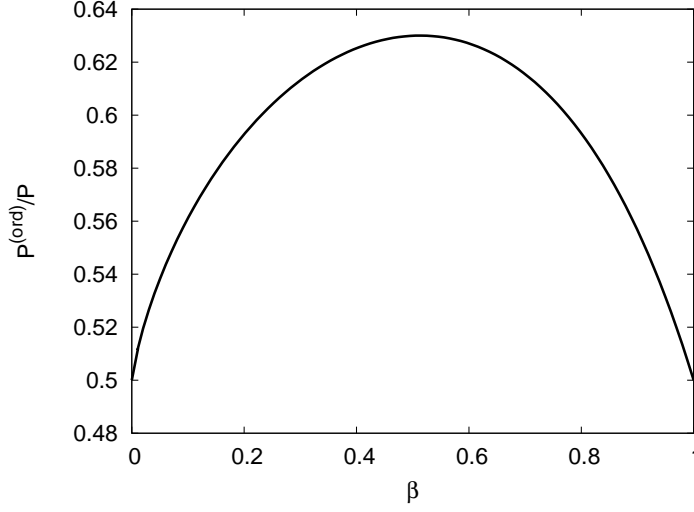


Fig. 3 The ratio of the pressures for the ordered and unconstrained charge systems versus the inverse temperature β .

For a fixed z , the ratio of this pressure to the pressure for the unconstrained charge system (2.12) turns out to be

$$\frac{P^{(\text{ord})}}{P} = 2 \sin \left(\frac{\pi\beta}{2(2-\beta)} \right) \left[\frac{1}{2 \sin \left(\frac{\pi\beta}{2} \right)} \right]^{\frac{2}{2-\beta}}. \quad (3.11)$$

This ratio is plotted as a function of the inverse temperature β in Fig. 3. It is seen that the pressure inequality in (3.7) is satisfied. Interestingly, the ratio takes the minimum value $1/2$ just at the limiting points $\beta \rightarrow 0$ and $\beta \rightarrow 1$.

In Appendix A, a fusion relation between the grand partition functions of the unconstrained and ordered log-gases is used to rederive in an alternative way the relationship between the corresponding bulk pressures (3.11).

Having the bulk pressure (3.10), the density-fugacity relationship reads as

$$\frac{n^{(\text{ord})}}{z^{\frac{2}{2-\beta}}} = \frac{2}{\sqrt{\pi}(2-\beta)} \frac{\Gamma\left(\frac{1-\beta}{2-\beta}\right)}{\Gamma\left(\frac{4-3\beta}{4-2\beta}\right)} \left[\Gamma\left(1 - \frac{\beta}{2}\right) \right]^{\frac{2}{2-\beta}}. \quad (3.12)$$

The small- β expansion of the rhs of this formula

$$\frac{n^{(\text{ord})}}{z^{\frac{2}{2-\beta}}} = 1 + \frac{1}{2}(C + 1 + \ln 2)\beta + O(\beta^2) \quad (3.13)$$

is analytic, which is in contrast to the singular result (2.16) for the unconstrained log-gas. This expansion is checked in Appendix B, using pair distributions of the free ($\beta = 0$) log-gas with alternating \pm charges [1]. For

a fixed z and in the limit $\beta \rightarrow 1^-$, the term $\Gamma((1-\beta)/(2-\beta)) \sim 1/(1-\beta)$ implies the collapse singularity

$$n^{(\text{ord})} \underset{\beta \rightarrow 1^-}{\sim} \frac{2z^2}{1-\beta}. \quad (3.14)$$

The adequacy of this term, in the sense that it compensates exactly the singularity of the hard-core contribution at the collapse point, will be documented in the next part. The plot of the density-fugacity relationship for ordered pointlike particles (3.12), valid up to $\beta_c = 1$, is represented in Fig. 4 by the solid curve. The curve is monotonously increasing in the whole β -interval $[0, 1]$, which is in contrast to Fig. 2 for the unconstrained system.

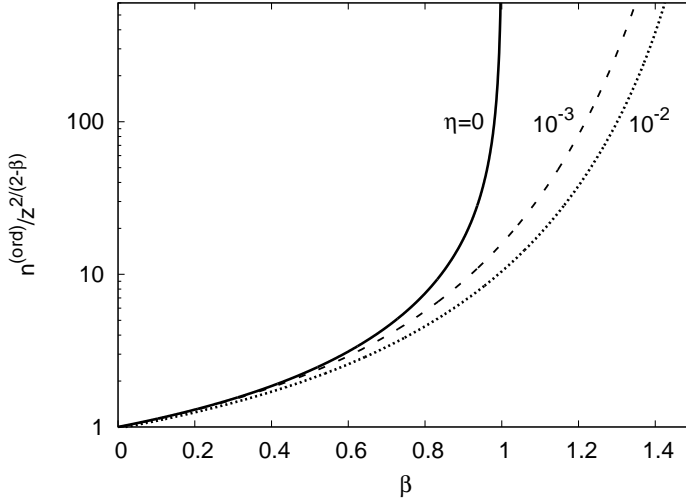


Fig. 4 The density-fugacity relationship versus the inverse temperature β for the ordered log-gas. The solid curve describes pointlike charges with the packing fraction $\eta = 0$. The dashed and dotted curves correspond to $\eta = 10^{-3}$ and 10^{-2} , respectively.

The virial equation of state takes the usual form

$$\beta P^{(\text{ord})} = \left(1 - \frac{\beta}{2}\right) n^{(\text{ord})}. \quad (3.15)$$

The (dimensionless) Helmholtz free energy per particle reads

$$\begin{aligned} f^{(\text{ord})}(\beta, n) = & \ln(\lambda n) - \frac{\beta}{2} \ln(r_0 n) + \frac{1}{2} \left(1 - \frac{\beta}{2}\right) \ln \pi \\ & - \left(1 - \frac{\beta}{2}\right) \left[1 - \ln\left(1 - \frac{\beta}{2}\right)\right] - \ln \Gamma\left(1 - \frac{\beta}{2}\right) \\ & - \left(1 - \frac{\beta}{2}\right) \left[\ln \Gamma\left(\frac{1-\beta}{2-\beta}\right) - \ln \Gamma\left(\frac{4-3\beta}{4-2\beta}\right)\right]. \end{aligned} \quad (3.16)$$

For the excess specific heat, we obtain

$$\begin{aligned} \frac{c_v^{(\text{ord})\text{ex}}}{k_B} &= \frac{\beta^2}{2(2-\beta)^3} \left[\psi' \left(\frac{1-\beta}{2-\beta} \right) - \psi' \left(\frac{4-3\beta}{4-2\beta} \right) \right] \\ &\quad - \frac{\beta^2}{2(2-\beta)} + \frac{\beta^2}{4} \psi' \left(1 - \frac{\beta}{2} \right). \end{aligned} \quad (3.17)$$

Close to the collapse point $\beta \rightarrow 1^-$, it exhibits the singular behavior

$$\frac{c_v^{(\text{ord})\text{ex}}}{k_B} = \frac{1}{2(1-\beta)^2} - \frac{3}{2(1-\beta)} + O(1), \quad (3.18)$$

which is exactly the same as in the unconstrained case, see Eq. (2.28).

3.2 Particles with hard cores

We attach to particles hard cores of diameter σ and derive the equation of state directly from the definition of the grand partition function, in close analogy with the procedure for the unconstrained system starting from Eq. (2.44).

For the ordered charge system of large size L , the grand partition function is defined by

$$\Xi_L^{(\text{ord})}(z, \sigma) = 1 + \sum_{N=1}^{\infty} z^{2N} I_{2N}^{(\text{ord})}(\sigma), \quad (3.19)$$

where the configuration integrals

$$\begin{aligned} I_{2N}^{(\text{ord})}(\sigma) &= \int_0^L dx_1 \int_0^{x_1} dx'_1 \int_0^{x'_1} dx_2 \int_0^{x_2} dx'_2 \cdots \int_0^{x'_{N-1}} dx_N \int_0^{x_N} dx'_N \\ &\quad \times B_{2N}(\{x_i\}, \{x'_i\}) \prod_{i=1}^N \theta(x_i - x'_i - \sigma) \prod_{i=1}^{N-1} \theta(x'_i - x_{i+1} - \sigma) \end{aligned} \quad (3.20)$$

involve the Boltzmann weights (2.48). As before, we substitute each Heaviside theta function in (3.20) via (2.50). For the lowest-order integral, we get

$$\begin{aligned} I_2^{(\text{ord})}(\sigma) &= I_2^{(\text{ord})}(0) - \int_0^L dx_1 \int_0^{x_1} dx'_1 \frac{1}{|x_1 - x'_1|^\beta} \theta(\sigma - x_1 + x'_1) \\ &\sim I_2(0) - L \int_0^\sigma dx x^{-\beta} = I_2(0) - L \frac{\sigma^{1-\beta}}{1-\beta}. \end{aligned} \quad (3.21)$$

For a general integral $I_{2N}^{(\text{ord})}$ in (3.20), we expand the product of theta functions written as in (2.50). The first term (unity) implies $I_{2N}^{(\text{ord})}(0)$. The second term contains $(2N-1)$ theta functions; each theta function forces two oppositely charged nearest-neighbor particles to be very close to one another for small σ , and therefore to form an almost neutral entity which decouples from all other charges. The integration over distance between these nearest

neighbors produces the factor $-\sigma^{1-\beta}/(1-\beta)$. The integration over the x -coordinate of the neutral entity, which exhibits all possible orderings with respect to a given configuration of other “active” charges, implies L . The remaining $2(N-1)$ charges contribute by $I_{2(N-1)}^{(\text{ord})}(0)$, so the total contribution is $[-L\sigma^{1-\beta}/(1-\beta)]I_{2(N-1)}^{(\text{ord})}(0)$. In the case of k neutral entities, there is an indistinguishability factor $1/k!$ attached to $[-L\sigma^{1-\beta}/(1-\beta)]^k I_{2(N-k)}^{(\text{ord})}(0)$. We end up with the recurrence relation

$$I_{2N}^{(\text{ord})}(\sigma) = I_{2N}^{(\text{ord})}(0) - \left(L \frac{\sigma^{1-\beta}}{1-\beta} \right) I_{2(N-1)}^{(\text{ord})}(0) + \frac{1}{2!} \left(L \frac{\sigma^{1-\beta}}{1-\beta} \right)^2 I_{2(N-2)}^{(\text{ord})}(0) + \dots \quad (3.22)$$

with $I_0^{(\text{ord})}(0) = 1$. Inserting this recurrence into the definition (3.19), we end up with

$$\Xi_L^{(\text{ord})}(z, \sigma) = \Xi_L^{(\text{ord})}(z, 0) \exp \left(-z^2 \frac{\sigma^{1-\beta}}{1-\beta} L \right). \quad (3.23)$$

Applying the logarithm to both sides of this equation and in the limit $L \rightarrow \infty$, we get the bulk pressure of the ordered system of charges with hard cores:

$$\beta P^{(\text{ord})}(z, \sigma) = \beta P^{(\text{ord})}(z, 0) - z^2 \frac{\sigma^{1-\beta}}{1-\beta}, \quad \xi \rightarrow 0, \quad (3.24)$$

where $\beta P^{(\text{ord})}(z, 0)$ is given by (3.10). The corresponding density-fugacity relationship reads as

$$n^{(\text{ord})}(z, \sigma) = n^{(\text{ord})}(z, 0) - 2z^2 \frac{\sigma^{1-\beta}}{1-\beta}, \quad \xi \rightarrow 0. \quad (3.25)$$

Although these results were derived in the stability region $0 \leq \beta < 1$, we assume that they can be analytically continued beyond the collapse point $\beta_c = 1$.

To provide a convenient representation of (3.25), we express the density-fugacity relation for pointlike particles (3.12) as follows

$$n^{(\text{ord})}(z, 0) = \frac{2\Phi^{(\text{ord})}(\beta)}{1-\beta} z^{\frac{2}{2-\beta}}, \quad (3.26)$$

where the function

$$\Phi^{(\text{ord})}(\beta) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{3-2\beta}{2-\beta}\right)}{\Gamma\left(\frac{4-3\beta}{4-2\beta}\right)} \left[\Gamma\left(1 - \frac{\beta}{2}\right) \right]^{\frac{2}{2-\beta}} \quad (3.27)$$

was chosen such that it equals to 1 at the collapse point, namely

$$\Phi^{(\text{ord})}(\beta) = 1 + (C + \ln \pi)(\beta - 1) + O((\beta - 1)^2). \quad (3.28)$$

Using the representation (3.26) in (3.25), we get the density-fugacity relationship which involves the leading hard-core correction:

$$\frac{n^{(\text{ord})}(z, \sigma)}{z^{\frac{2}{2-\beta}}} = \frac{2}{1-\beta} \left[\Phi^{(\text{ord})}(\beta) - \xi^{1-\beta} \right], \quad \xi \rightarrow 0. \quad (3.29)$$

The leading correction $\propto \xi^{1-\beta}$ makes the density finite at the collapse point $\beta_c = 1$,

$$\frac{n^{(\text{ord})}(z, \sigma)}{z^2} = -2 (\ln \xi + C + \ln \pi). \quad (3.30)$$

The formalism holds up to $\beta = 3/2$ at which $\Phi^{(\text{ord})}(\beta)$ diverges.

It is convenient to pass from the parameter $\xi = \sigma z^{\frac{2}{2-\beta}}$ to the packing fraction $\eta = n^{(\text{ord})} \sigma$. Denoting $g \equiv n^{(\text{ord})}(z, \sigma) / z^{\frac{2}{2-\beta}}$, we have $\eta = g\xi$. Multiplying both sides of Eq. (3.29) by $(1-\beta)g^{1-\beta}$, we get

$$(1-\beta)g^{2-\beta} - 2 \left[\Phi^{(\text{ord})}(\beta)g^{1-\beta} - \eta^{1-\beta} \right] = 0. \quad (3.31)$$

At $\beta = 0$, we take the solution $g = (1 + \sqrt{1-8\eta})/2$ and follow this branch with increasing β . The density-fugacity relation for the packing fractions 10^{-3} and 10^{-2} are plotted in Fig. 4 by the dashed and dotted curves, respectively. The functions are monotonously increasing in the whole interval of β , which is in contrast to the unconstrained system (see Fig. 2).

4 Conclusion

The aim of this work was to derive and analyze the exact thermodynamics of the two-component log-gas formulated on an infinite line with alternating \pm charges.

The thermodynamics was first solved for the case of pointlike particles, up to the collapse point $\beta_c = 1$, by using the TBA results. The obtained density-fugacity relationship (3.12) turns out to be analytic in the high-temperature region, see the small- β expansion (3.13) and its check in the Appendix B. Moreover, as is shown in Fig. 4 by the solid curve, the density-fugacity ratio $n^{(\text{ord})}/z^{2/(2-\beta)}$ is a monotonously increasing function of the inverse temperature β . This is in contrast to the unconstrained log-gas with a non-analytic small- β expansion and a non-monotonous plot of $n/z^{2/(2-\beta)}$ versus β , see the solid curve in Fig. 2. For a fixed fugacity z , the validity of the inequality (3.7) for the pressures of the unconstrained and ordered log-gases was confirmed by finding the exact ratio $P^{(\text{ord})}/P$ (Fig. 3); the minimum $1/2$ is reached just at the limiting points $\beta \rightarrow 0$ and $\beta \rightarrow 1$.

The inclusion of hard cores around the particles was a more complicated task than in the unconstrained case since we could not apply the screening sum rule for pair correlation functions valid for a conductor. The derivation of the leading hard-core correction to the pressure, outlined in Sect. 3.2, was based on the explicit definition of the grand partition function and a substitution of Heaviside theta functions due to the presence of hard cores.

The final result for the bulk pressure (3.24) and the corresponding density-fugacity relationship (3.25) pass the collapse test, in the sense that the hard-core correction compensates exactly the singularity of the pointlike model and leads to a finite pressure and particle density at $\beta = 1$.

Since the log-gas is by the structure much simpler than the 2D Coulomb gas, it might be possible to get all relevant hard-core corrections which compensate an infinite series of singularities of the pointlike result and allow us to go up to $\beta_{KT} = 2$, in the spirit of Refs. [23, 31, 32]. This problem is left for future.

Appendix A

Here, we rederive in an alternative way the relationship between the bulk pressures of the ordered and unconstrained log-gases (3.11).

We use the fusion relation between the grand partition functions of the unconstrained and ordered log-gases [12, 13]:

$$\Xi_L^{(\text{ord})}((q - q^{-1})z) = \frac{\Xi_L(qz) + \Xi_L(q^{-1}z)}{2\Xi_L(z)}, \quad q = e^{i\pi\beta/2}. \quad (\text{A.1})$$

We know that for large L the grand partition functions behave as

$$\Xi_L^{(\text{ord})}(z) \underset{L \rightarrow \infty}{\sim} \exp[\beta P^{(\text{ord})}(z)L], \quad \beta P^{(\text{ord})}(z) = a^{(\text{ord})}(\beta)z^{\frac{2}{2-\beta}} \quad (\text{A.2})$$

and

$$\Xi_L(z) \underset{L \rightarrow \infty}{\sim} \exp[\beta P(z)L], \quad \beta P(z) = a(\beta)z^{\frac{2}{2-\beta}}. \quad (\text{A.3})$$

The lhs of Eq. (A.1) reads

$$\begin{aligned} & \exp \left\{ -La^{(\text{ord})}(\beta) \left[2z \sin \left(\frac{\pi\beta}{2} \right) \right]^{\frac{2}{2-\beta}} \sin \left(\frac{\pi\beta}{2(2-\beta)} \right) \right\} \\ & \times \cos \left\{ La^{(\text{ord})}(\beta) \left[2z \sin \left(\frac{\pi\beta}{2} \right) \right]^{\frac{2}{2-\beta}} \cos \left(\frac{\pi\beta}{2(2-\beta)} \right) \right\}. \end{aligned} \quad (\text{A.4})$$

The rhs of Eq. (A.1) can be expressed as

$$\exp \left\{ La(\beta)z^{\frac{2}{2-\beta}} \left[\cos \left(\frac{\pi\beta}{2-\beta} \right) - 1 \right] \right\} \cos \left[La(\beta)z^{\frac{2}{2-\beta}} \sin \left(\frac{\pi\beta}{2-\beta} \right) \right]. \quad (\text{A.5})$$

The equality of separately exponential and cosine terms in Eqs. (A.4) and (A.5) leads to the only relation

$$\frac{a^{(\text{ord})}(\beta)}{a(\beta)} = \frac{P^{(\text{ord})}}{P} = 2 \sin \left(\frac{\pi\beta}{2(2-\beta)} \right) \left[\frac{1}{2 \sin \left(\frac{\pi\beta}{2} \right)} \right]^{\frac{2}{2-\beta}}, \quad (\text{A.6})$$

in agreement with (3.11).

Appendix B

Here, we construct the small- β expansion of the pressure and the particle density for the ordered log-gas.

For $\beta = 0$, the Boltzmann factor $B_{2N}(\{\varphi_i\}, \{\varphi'_i\}) = 1$. The ordered integrals (3.3) are simply given by $I_{2N}^{(\text{ord})} = 1/(2N)!$ as there are just $(2N)!$ ways how to order the angle variables and each ordering implies the same contribution. This leads to

$$\Xi_L^{(\text{ord})} = 1 + \sum_{N=1}^{\infty} \frac{(zL)^{2N}}{(2N)!} \underset{L \rightarrow \infty}{\sim} \frac{1}{2} \exp(zL). \quad (\text{B.1})$$

Consequently, in the lowest order,

$$\beta P_0^{(\text{ord})}(z) = z, \quad n_0^{(\text{ord})}(z) = z. \quad (\text{B.2})$$

For a given configuration of particles $\{j\}$ with charges $\{q_j\}$ at positions $\{x_j\}$, the interaction energy is given by

$$\begin{aligned} E &= \frac{1}{2} \sum_{j \neq k} q_j q_k (-\ln |x_j - x_k|) \\ &= \frac{1}{2} \int_0^L dx \int_0^L dx' (-\ln |x - x'|) \sum_{q, q' = \pm 1} qq' \hat{n}_{qq'}(x, x'), \end{aligned} \quad (\text{B.3})$$

where the microscopic quantity

$$\hat{n}_{qq'}(x, x') = \sum_{j \neq k} \delta_{q, q_j} \delta(x - x_j) \delta_{q', q_k} \delta(x' - x_k). \quad (\text{B.4})$$

Expanding the Boltzmann factor in β , $e^{-\beta E} \sim 1 - \beta E$, and using the cumulant expansion, we get the leading β -correction to the pressure,

$$\beta P_1^{(\text{ord})}(z) = \beta P_0^{(\text{ord})}(z) - \beta \frac{\langle E \rangle_0}{L}, \quad (\text{B.5})$$

where the thermal averaging $\langle \cdots \rangle_0$ is over the system of non-interacting ($\beta = 0$) ordered charges. For an infinite system $L \rightarrow \infty$, the two-body distributions $n_{qq'}(x, x') \equiv n_{qq'}(|x - x'|) = \langle \hat{n}_{qq'}(x, x') \rangle_0$ have been calculated in [1]:

$$n_{qq}(x) = \left(\frac{n}{2}\right)^2 \left(1 - e^{-2n|x|}\right), \quad n_{q,-q}(x) = \left(\frac{n}{2}\right)^2 \left(1 + e^{-2n|x|}\right). \quad (\text{B.6})$$

Considering the lowest order for n from (B.2), we obtain

$$\begin{aligned} \beta P_1^{(\text{ord})}(z) &= z + \frac{\beta}{2} \int_{-\infty}^{\infty} dx \ln |x| [n_{++}(x) + n_{--}(x) - n_{+-}(x) - n_{-+}(x)] \\ &= z + \frac{\beta}{2} z (C + \ln 2) + \frac{\beta}{2} z \ln z. \end{aligned} \quad (\text{B.7})$$

The corresponding density of particles

$$n_1^{(\text{ord})}(z) = z + \frac{\beta}{2}z(C + 1 + \ln 2) + \frac{\beta}{2}z \ln z \quad (\text{B.8})$$

is consistent with the small- β expansion (3.13), where $z^{\frac{2}{2-\beta}} \sim z[1 + (\beta/2) \ln z]$.

To find the next order of the small- β expansion we need to know four-body distributions for the system of non-interacting ordered charges, and so on.

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